University of Alabama in Huntsville
Department of Physics
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PH 305: Mathematical Methods in Physics

Homework Solutions No. 1

Solution 1:

1. The viscosity μ is the proportionality coefficient in the given expression
$$F \sim \mu \frac{\Delta v}{\Delta z} S\,.$$
Therefore,
$$\mu \sim \frac{F}{S\,(\,\Delta v/\,\Delta z)} \sim \frac{[MLT^{-2}]}{[L^2]\cdot[T^{-1}]} \sim [ML^{-1}T^{-1}]\,.$$

2. The flow rate satisfies $\Phi = V/t$, where V is the volume of flow during time t. Hence, $\Phi \sim [L^3T^{-1}]$. According to the Buckingham "pi" theorem, we are looking for a combination of the form
$$\Phi \left(\frac{\partial p}{\partial x}\right)^{\alpha} \mu^{\beta}\, R^{\gamma} \sim [1]\,,$$
that is
$$[L^3T^{-1}]\,[ML^{-2}T^{-2}]^{\alpha}\,[ML^{-1}T^{-1}]^{\beta}\,L^{\gamma} \sim 1\,.$$
This gives us the equations to solve:
$$\begin{aligned} 3-2\alpha-\beta+\gamma &= 0 \\ \alpha+\beta &= 0 \\ -1-2\alpha-\beta &= 0 \end{aligned}$$
whose solution is
$$\begin{aligned} \alpha &= -1 \\ \beta &= 1 \\ \gamma &= -4\,. \end{aligned}$$

Using the "pi" theorem, we then find

$$\Phi \sim C \, \frac{1}{\mu} \left(\frac{\partial p}{\partial x} \right) R^4 \, .$$

Here, C is a dimensionless constant that characterizes the flow. There is just one such dimensionless constant, as according to the "pi" theorem

$$\Pi = N - P = 4 - 3 = 1 \, ,$$

where N is the number of variables describing the system $(\Phi, \mu, R, \, \partial p / \, \partial x)$, and P is the number of physical dimensions (M, L, T).

3. We expect the flow rate to equal the cross sectional area of the pipe times the velocity of the fluid, i.e., $\Phi = Sv$. The cross sectional area $S = \pi R^2$, so that our result means that the velocity profile (i.e., the dependence of the velocity as a function of the distance from the cylinder's axis, z, must be parabolic, or $v(z) \sim z^2$. The boundary condition that viscous fluid flow satisfies is that there is zero relative motion between the fluid and the walls of the pipe on the boundary, so that $v(z = R) = 0$. The flow at the cylinder's axis has to be higher than closer to the wall, and the dependence has to be parabolic. Therefore, we expect $v(z) \sim R^2 - z^2$. This means that the dimensional analysis dependence on R^4 may be refined to $R^2(R^2 - z^2)$, as this is the only way to get the right boundary condition. The dependence on the viscosity and pressure gradient is much simpler: the greater the pressure gradient the greater the flow rate, because it increases the flow velocity, and the greater the viscosity, the smaller the flow rate, as the effect of the boundary condition is felt more and more towards the center of the cylinder, and the flow velocity is decreasing. We could therefore conclude that dimensional analysis plus the boundary condition gives us the result
$$\Phi \sim C \, \frac{1}{\mu} \left(\frac{\partial p}{\partial x} \right) R^2 \, (R^2 - z^2) \, .$$

4. First, let us quote the solution of the Navier–Stokes equation:
$$\Phi = \frac{\pi}{4} \frac{1}{\mu} \left(\frac{\partial p}{\partial x} \right) R^2 \, (R^2 - z^2) \, .$$

We see that the Navier–Stokes equations can give us the value of the dimensionless constant C that characterizes the flow, specifically that $C = \pi/4$. We cannot find this by dimensional analysis (augmented by boundary conditions) alone. The value of C depends on the shape of the pipe. For example, if the pipe is rectangular, $C = 1/2$. To find the value of C we must solve the fluid flow equations in their full glory. But this is much harder than doing dimensional analysis, and for many applications we don't have to know the value of C and the functional form is sufficient. Also, it is very useful, before we start solving the differential equations, to already have an idea what the solution looks like. Dimensional analysis is an invaluable tool for that purpose.